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S-asymptotically ω -periodic solution for fractional differential equations of order $q \in (0, 1)$ with finite delay

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Abstract

In this paper, we investigate the existence and uniqueness of S-asymptotically ω -periodic solutions to fractional differential equations of order $q \in (0, 1)$ with finite delay in a Banach space X . Existence and uniqueness theorems, which are new even in the case of $X = \mathbf{R}^n$ or $A = 0$, are established. As examples of applications of our existence and uniqueness results, we obtain the S-asymptotically ω -periodic solutions for the fractional-order autonomous neural networks with delay.

Keywords: ω -periodic; S-asymptotically ω -periodic; Caputo derivative; fractional differential equation; finite delay

1 Introduction

As one important branch of the research on evolution equations, the study of fractional differential equations in Banach spaces is very active recently due to its strong background in physics, chemistry, engineering, biology, financial sciences, etc. (cf., e.g., [1–18] and the references therein). In this paper, we are concerned with the following fractional differential equation:

$$\begin{cases} {}^c D_t^q u(t) = Au(t) + f(t, u_t), & t \geq 0, \\ u(t) = \phi(t), & t \in [-\delta, 0], \end{cases} \quad (1.1)$$

where $\delta > 0$, $q \in (0, 1)$ and the fractional derivative is understood here in the Caputo sense, $A : D(A) \subset X \rightarrow X$ is the generator of an analytic semigroup on a Banach space X , f is a given function, $u_t : [-\delta, 0] \rightarrow X$ is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-\delta, 0]$ (cf., e.g., [19–23]), and $\phi \in C([-\delta, 0], X)$. Our main purpose is to establish existence and uniqueness theorems about the S-asymptotically ω -periodic solutions to the (1.1).

Actually, while the almost periodic, almost automorphic, and weighted pseudo almost periodic solutions to various evolution equations are investigated by many scholars (cf., e.g., [1, 22–29]), the S-asymptotically ω -periodic solutions to some evolution equations are also studied by some researchers. There have been several interesting contributions to the investigation of S-asymptotically ω -periodic solutions of differential equations and fractional differential equations in finite as well as infinite dimensional spaces (cf. [1–3,

5, 19, 28, 30]). We also note that some papers about the existence of S -asymptotically ω -periodic solutions of fractional differential equations focus on the order $q \in (1, 2)$ ([2, 5] and references therein). Therefore, motivated by all this work, we pay attention in this paper to the study of the existence of S -asymptotically ω -periodic (mild) solutions for differential equation of fractional order of type (1.1) for the case $q \in (0, 1)$.

The paper is organized as follows. In Section 2, we recall some basic notations and concepts. In Section 3, we discuss the existence and uniqueness of S -asymptotically ω -periodic mild solution, and as a special case of our result, we present the corresponding result in the case of $A = 0$ (Theorem 3.7). In Section 4, we apply our result to a study of the existence and uniqueness of S -asymptotically ω -periodic solution for the fractional-order neural network with finite delay.

2 Basic notations and concepts

Throughout this paper, $(X, \|\cdot\|)$ is a Banach space, $C_b(\mathbf{R}_+, X)$ denotes the space of the continuous bounded functions from $[0, +\infty)$ to X , endowed with the norm

$$\|f\|_\infty = \sup_{t \geq 0} \|f(t)\|.$$

$C([-\delta, 0], X)$ denotes the space of the continuous functions from $[-\delta, 0]$ to X with the norm

$$\|x\|_{[-\delta, 0]} = \sup_{t \in [-\delta, 0]} \|x(t)\|.$$

Definition 2.1 (Cf., e.g., [11, 15]) The fractional integral of order q with the lower limit zero for a function $f \in L^1[0, \infty)$ is defined as

$$I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad t > 0, 0 < q < 1,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (Cf., e.g., [11, 15]) The Caputo derivative of order q for a function $f \in C^1[0, \infty)$ can be written as

$${}^c D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} ds, \quad t > 0, 0 < q < 1.$$

3 Existence and uniqueness theorems

In this section we discuss the existence and uniqueness of S -asymptotically ω -periodic solutions for problem (1.1).

Let A be the infinitesimal generator of a uniformly exponentially stable analytic semigroup of linear operators $\{T(t)\}_{t \geq 0}$ on X such that

$$\|T(t)\| \leq M e^{-\mu t}, \quad t \geq 0,$$

where $M, \mu > 0$ are constants.

Based on the work in [6, 12], we define the mild solution for problem (1.1) as follows.

Definition 3.1 A function $u \in C([- \delta, +\infty], X)$ satisfying the equation

$$u(t) = \begin{cases} \phi(t), & t \in [-\delta, 0], \\ Q(t)\phi(0) + \int_0^t (t-s)^{q-1} R(t-s)f(s, u_s) ds, & t > 0, \end{cases} \quad (3.1)$$

is called a mild solution of problem (1.1), where

$$Q(t) = \int_0^\infty \xi_q(\sigma) T(t^q \sigma) d\sigma, \quad R(t) = q \int_0^\infty \sigma \xi_q(\sigma) T(t^q \sigma) d\sigma.$$

Here ξ_q is a probability density function defined on $(0, \infty)$ (see [12]) such that

$$\xi_q(\sigma) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-\sigma)^{n-1} \frac{\Gamma(nq)}{(n-1)!} \sin(n\pi q) \geq 0, \quad \sigma \in (0, \infty).$$

In fact, we can see that $\xi_q(\sigma)$ is the Wright type function in [11, 15]. For $-1 < r < \infty$, the following conclusions hold.

$$(A1) \quad \int_0^\infty \sigma^r \xi_q(\sigma) d\sigma = \frac{\Gamma(1+r)}{\Gamma(1+qr)};$$

$$(A2) \quad \int_0^\infty \xi_q(\sigma) e^{-z\sigma} d\sigma = E_q(-z), \quad z \in \mathbf{C};$$

$$(A3) \quad \int_0^\infty q\sigma \xi_q(\sigma) e^{-z\sigma} d\sigma = E_{q,q}(-z), \quad z \in \mathbf{C},$$

where $E_q(\cdot)$ ($E_{q,q}(\cdot)$) is the Mittag-Leffler function (the generalized Mittag-Leffler function) (cf., e.g., [11, 15]).

Remark 3.2

(i) Noting that $\int_0^\infty \xi_q(\sigma) d\sigma = 1$, we get

$$\|Q(t)\| \leq M \quad \text{and} \quad \lim_{t \rightarrow \infty} \|Q(t)\| = 0. \quad (3.2)$$

(ii) In view of (A1), we have

$$\|R(t)\| \leq \frac{M}{\Gamma(q)}, \quad t \geq 0, \quad (3.3)$$

$$\begin{aligned} \int_0^t (t-s)^{q-1} \|R(t-s)\| ds &\leq qM \int_0^t \int_0^\infty \sigma \xi_q(\sigma) (t-s)^{q-1} e^{-\mu(t-s)^q \sigma} d\sigma ds \\ &\leq M \int_0^\infty \xi_q(\sigma) d\sigma \int_0^\infty e^{-\mu\tau} d\tau \\ &= \frac{M}{\mu}. \end{aligned} \quad (3.4)$$

(iii) If $A \in \mathbf{R}^{n \times n}$ is a constant matrix, then A generates a bounded operator semigroup $T(t) = e^{At}$ on X . Hence

$$\begin{aligned} Q(t) &= \int_0^\infty \xi_q(\sigma) e^{At^q \sigma} d\sigma = E_q(At^q), \\ R(t) &= q \int_0^\infty \sigma \xi_q(\sigma) e^{At^q \sigma} d\sigma = E_{q,q}(At^q). \end{aligned}$$

It follows from Definition 3.1 that

$$u(t) = \begin{cases} \phi(t), & t \in [-\delta, 0], \\ E_q(At^q)\phi(0) + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s, u_s) ds, & t > 0, \end{cases}$$

is a mild solution of the following problem:

$$\begin{cases} {}^c D_t^q u(t) = Au(t) + f(t, u_t), & t \geq 0, \\ u(t) = \phi(t), & t \in [-\delta, 0]. \end{cases} \quad (3.5)$$

It is not difficult to see that $u(t)$ actually is the solution for the problem (3.5).

The following definition of S-asymptotically ω -periodic functions taking values in a Banach space X is from [28].

Definition 3.3 A function $h \in C_b(\mathbf{R}_+, X)$ is called S-asymptotically ω -periodic if there exists $\omega > 0$ such that $\lim_{t \rightarrow \infty} (h(t + \omega) - h(t)) = 0$. In this case, we say that ω is an asymptotic period of h .

Let $\text{SAP}_\omega(X)$ represent the space of all the X -valued S-asymptotically ω -periodic functions endowed with the uniform convergence norm denoted by $\|\cdot\|_\infty$. Then, by virtue of [28], Proposition 3.5, $\text{SAP}_\omega(X)$ is a Banach space.

Set

$$\text{SAP}_{\omega,0}(X) = \{x \in \text{SAP}_\omega(X) : x(0) = 0\}.$$

Clearly, $\text{SAP}_{\omega,0}(X)$ is a closed subspace of $\text{SAP}_\omega(X)$.

Lemma 3.4 Let $u : [-\delta, +\infty) \rightarrow X$ be a function with $u_0 \in C([-\delta, 0], X)$ and $u|_{[0, +\infty)} \in \text{SAP}_\omega(X)$. Then the function $t \rightarrow u_t$ belongs to $\text{SAP}_\omega(C([-\delta, 0], X))$.

Proof Since u_t is continuous on $[-\delta, 0]$, we see that there exists $\bar{\theta} \in [-\delta, 0]$ such that

$$\|u_{t+\omega} - u_t\|_{[-\delta, 0]} = \sup_{-\delta \leq \theta \leq 0} \|u(t + \omega + \theta) - u(t + \theta)\| = \|u(t + \omega + \bar{\theta}) - u(t + \bar{\theta})\|.$$

Setting $\tau = t + \bar{\theta}$, we obtain

$$\lim_{t \rightarrow +\infty} \|u(t + \omega + \bar{\theta}) - u(t + \bar{\theta})\| = \lim_{\tau \rightarrow +\infty} \|u(\tau + \omega) - u(\tau)\| = 0. \quad \square$$

Set

$$\tilde{C}_b(X) = \{x \in C_b([-\delta, +\infty), X) : x|_{t \geq 0} \in C_b(\mathbf{R}_+, X), x|_{[-\delta, 0]} = 0\}.$$

For the function $f : \mathbf{R}_+ \times C([-\delta, 0], X) \rightarrow X$, we write

(H1) there exists a function $s \rightarrow L_f(s) \in L^1([0, t], \mathbf{R}_+)$ such that

$$\|f(t, \psi_1) - f(t, \psi_2)\| \leq L_f(t) \|\psi_1 - \psi_2\|_{[-\delta, 0]}, \quad \text{for all } t \geq 0, \psi_1, \psi_2 \in C([-\delta, 0], X),$$

the function $s \rightarrow \frac{L_f(s)}{(t-s)^{1-q}}$ belongs to $L^1([0, t], \mathbf{R}_+)$ and

$$\Lambda := \sup_{t \geq 0} \int_0^t \frac{L_f(s)}{(t-s)^{1-q}} ds < \frac{\Gamma(q)}{M}; \quad (3.6)$$

(H2) $K := \sup_{t \geq 0} \int_0^t \frac{\|f(s, 0)\|}{(t-s)^{1-q}} ds < \infty$;

(H3) there exists $\omega > 0$, for all $\varphi \in C([-\delta, 0], X)$, $\lim_{t \rightarrow \infty} \|f(t + \omega, \varphi) - f(t, \varphi)\| = 0$.

Theorem 3.5 Assume that (H1)-(H3) hold. Then the problem (1.1) has a unique S -asymptotically ω -periodic mild solution.

Proof For every $\phi \in C([-\delta, 0], X)$, we define the function $y(t) = \phi(t)$ for $t \in [-\delta, 0]$, $y(t) = Q(t)\phi(0)$ for $t \geq 0$. Then $y \in C([-\delta, \infty), X)$. Set

$$u(t) = x(t) + y(t), \quad t \in [-\delta, +\infty).$$

It is obvious that u satisfies (3.1) if and only if x satisfies $x_0 = 0$ and for $t \geq 0$,

$$x(t) = \int_0^t (t-s)^{q-1} R(t-s) f(s, x_s + y_s) ds.$$

For each $x \in \tilde{C}_b(X)$, we write $C_1 = \|x\|_\infty + M\|\phi(0)\| + \|\phi\|_{[-\delta, 0]}$. Then

$$\begin{aligned} \|x_t + y_t\|_{[-\delta, 0]} &\leq \sup_{-\delta \leq \theta \leq 0} \|x(t + \theta)\| + \sup_{-\delta \leq \theta \leq 0} \|y(t + \theta)\| \\ &\leq \sup_{0 \leq \tau \leq t} \|x(\tau)\| + M\|\phi(0)\| + \|\phi\|_{[-\delta, 0]} \\ &\leq C_1. \end{aligned}$$

Hence

$$\|f(t, x_t + y_t)\| \leq L_f(t) \|x_t + y_t\|_{[-\delta, 0]} + \|f(t, 0)\| \leq C_1 L_f(t) + \|f(t, 0)\|. \quad (3.7)$$

We consider the operator $\mathcal{F} : \tilde{C}_b(X) \rightarrow \tilde{C}_b(X)$ as follows:

$$(\mathcal{F}x)(t) = \begin{cases} 0, & t \in [-\delta, 0], \\ \int_0^t (t-s)^{q-1} R(t-s) f(s, x_s + y_s) ds, & t \geq 0. \end{cases}$$

In view of (3.3), (3.7), (3.6) and (H2), we have

$$\left\| \int_0^t \frac{R(t-s) f(s, x_s + y_s)}{(t-s)^{1-q}} ds \right\| \leq \frac{MC_1}{\Gamma(q)} \int_0^t \frac{L_f(s)}{(t-s)^{1-q}} ds + \frac{MK}{\Gamma(q)} < C_1 + \frac{MK}{\Gamma(q)}. \quad (3.8)$$

So, the operator \mathcal{F} is well defined.

It is clear that the fixed points of \mathcal{F} are mild solutions to problem (1.1).

Now, we show that \mathcal{F} is $\text{SAP}_{\omega,0}(X)$ -valued.

For each $x \in \text{SAP}_{\omega,0}(X)$, (3.2) implies that $y|_{[0,\infty)} \in \text{SAP}_{\omega}(X)$. It follows from Lemma 3.4 that the function $t \rightarrow y_t$ belongs to $\text{SAP}_{\omega}(C([- \delta, 0], X))$.

Moreover, we have

$$\begin{aligned} & \|(\mathcal{F}x)(t+\omega) - (\mathcal{F}x)(t)\| \\ &= \left\| \int_0^{\omega} \frac{R(t+\omega-s)}{(t+\omega-s)^{1-q}} f(s, x_s + y_s) ds + \int_{\omega}^{t+\omega} \frac{R(t+\omega-s)}{(t+\omega-s)^{1-q}} f(s, x_s + y_s) ds \right. \\ &\quad \left. - \int_0^t (t-s)^{q-1} R(t-s) f(s, x_s + y_s) ds \right\| \\ &\leq \frac{M}{\Gamma(q)} \left[\int_0^{\omega} (t+\omega-s)^{q-1} (C_1 L_f(s) + \|f(s, 0)\|) ds \right] \\ &\quad + \int_0^t (t-s)^{q-1} \|R(t-s)\| \|f(s+\omega, x_{s+\omega} + y_{s+\omega}) - f(s, x_s + y_s)\| ds \\ &\quad + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, x_{s+\omega} + y_{s+\omega}) - f(s, x_s + y_s)\| ds \\ &= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

Noting that $t+\omega-s \geq \frac{t+\omega}{\omega}(\omega-s)$, we have

$$\int_0^{\omega} \frac{C_1 L_f(s) + \|f(s, 0)\|}{(t+\omega-s)^{1-q}} ds \leq \left(\frac{\omega}{t+\omega} \right)^{1-q} \int_0^{\omega} \frac{C_1 L_f(s) + \|f(s, 0)\|}{(\omega-s)^{1-q}} ds,$$

which implies that

$$I_1(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

By (3.7), we get

$$\begin{aligned} & \|f(s+\omega, x_{s+\omega} + y_{s+\omega}) - f(s, x_{s+\omega} + y_{s+\omega})\| \\ &\leq C_1 [L_f(s+\omega) + L_f(s)] + \|f(s+\omega, 0)\| + \|f(s, 0)\|. \end{aligned}$$

By (H3), we can see that, for each $\varepsilon > 0$, there is a positive constant L_1 such that

$$\|f(s+\omega, x_{s+\omega} + y_{s+\omega}) - f(s, x_{s+\omega} + y_{s+\omega})\| < \varepsilon, \quad s \geq L_1.$$

Noting that (3.4) and $t-s \geq \frac{t}{L_1}(L_1-s)$, we deduce that

$$\begin{aligned} I_2(t) &\leq \frac{M}{\Gamma(q)} \int_0^{L_1} \frac{C_1 L_f(s+\omega) + \|f(s+\omega, 0)\| + C_1 L_f(s) + \|f(s, 0)\|}{(t-s)^{1-q}} ds \\ &\quad + \varepsilon \int_{L_1}^t (t-s)^{q-1} \|R(t-s)\| ds \\ &\leq \frac{M}{\Gamma(q)} \left[\int_{\omega}^{L_1+\omega} \frac{C_1 L_f(s) + \|f(s, 0)\|}{(t+\omega-s)^{1-q}} ds + \int_0^{L_1} \frac{C_1 L_f(s) + \|f(s, 0)\|}{(t-s)^{1-q}} ds \right] + \frac{M\varepsilon}{\mu} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{M}{\Gamma(q)} \left(\frac{L_1 + \omega}{t} \right)^{1-q} \left[\int_0^{L_1+\omega} \frac{(C_1 L_f(s) + \|f(s, 0)\|) ds}{(L_1 + \omega - s)^{1-q}} \right] \\
&\quad + \frac{M}{\Gamma(q)} \left(\frac{L_1}{t} \right)^{1-q} \left[\int_0^{L_1} \frac{(C_1 L_f(s) + \|f(s, 0)\|) ds}{(L_1 - s)^{1-q}} \right] + \frac{M\varepsilon}{\mu} \\
&\leq \frac{2M}{\Gamma(q)} \left(\frac{L_1 + \omega}{t} \right)^{1-q} [C_1 \Lambda + K] + \frac{M\varepsilon}{\mu}.
\end{aligned} \tag{3.9}$$

Since

$$x_t + y_t \in \text{SAP}_\omega(C([- \delta, 0], X)),$$

we know that there is a positive constant $L_2 > 0$ such that

$$\|(x_{s+\omega} + y_{s+\omega}) - (x_s + y_s)\|_{[-\delta, 0]} \leq \varepsilon, \quad s \geq L_2,$$

then

$$\begin{aligned}
I_3(t) &\leq \frac{2M}{\Gamma(q)} \left[C_1 \int_0^{L_2} \frac{L_f(s)}{(t-s)^{1-q}} ds + \int_0^{L_2} \frac{\|f(s, 0)\|}{(t-s)^{1-q}} ds \right] + \frac{M\varepsilon}{\Gamma(q)} \int_{L_2}^t \frac{L_f(s)}{(t-s)^{1-q}} ds \\
&\leq \frac{M}{\Gamma(q)} \left[(2C_1 \Lambda + 2K) \left(\frac{L_2}{t} \right)^{1-q} + \Lambda \varepsilon \right].
\end{aligned}$$

Thus,

$$\|(\mathcal{F}x)(t + \omega) - (\mathcal{F}x)(t)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ and } \varepsilon \rightarrow 0.$$

So

$$\mathcal{F}(\text{SAP}_{\omega,0}(X)) \subseteq \text{SAP}_{\omega,0}(X).$$

Moreover, for $x, \tilde{x} \in \text{SAP}_{\omega,0}(X)$, we have

$$\|(\mathcal{F}x)(t) - (\mathcal{F}\tilde{x})(t)\| \leq \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} L_f(s) \|x_s - \tilde{x}_s\|_{[-\delta, 0]} ds \leq \frac{M\Lambda}{\Gamma(q)} \|x - \tilde{x}\|_\infty.$$

Hence

$$\|(\mathcal{F}x)(t) - (\mathcal{F}\tilde{x})(t)\|_\infty \leq \frac{M\Lambda}{\Gamma(q)} \|x - \tilde{x}\|_\infty,$$

which means that \mathcal{F} is a contraction mapping. Then the proof now can be finished by using the contraction mapping principle. \square

Remark 3.6 In the proof of Theorem 3.5, we can see the result is true when (H3) changes to

$$(H3') : \quad \lim_{t \rightarrow \infty} \int_0^t (t-s)^{q-1} \|f(s + \omega, \varphi) - f(s, \varphi)\| ds = 0,$$

because now we can directly obtain $\lim_{t \rightarrow \infty} I_2(t) = 0$.

In the case of $A \equiv 0$, (1.1) takes the form of

$$\begin{cases} {}^c D_t^q u(t) = f(t, u_t), & t \geq 0, \\ u(t) = \phi(t), & t \in [-\delta, 0]. \end{cases} \quad (3.10)$$

It is well known that (3.10) is equivalent to the following integral equation:

$$u(t) = \begin{cases} \phi(t), & t \in [-\delta, 0], \\ \phi(0) + \int_0^t (t-s)^{q-1} f(s, u_s) ds, & t \geq 0. \end{cases}$$

Thus, we study (3.10) just like the case of

$$Q(t) = R(t) \equiv I$$

in (3.1). Clearly, we just need to revise (3.8) and (3.9), that is, if we replace (H2), (H3) by

$$(H2') : \quad K := \sup_{t \geq 0} \left\| \int_0^t \frac{f(s, 0)}{(t-s)^{1-q}} ds \right\| < \infty$$

and (H3'), respectively, then we can obtain the corresponding result of problem (3.10), hence we have the following result.

Theorem 3.7 *Assume that (H1) ($M \equiv 1$), (H2') and (H3') hold. Then the following fractional differential equation:*

$$\begin{cases} {}^c D_t^q u(t) = f(t, u_t), & t \geq 0, \\ u(t) = \phi(t), & t \in [-\delta, 0], \end{cases}$$

has a unique S -asymptotically ω -periodic solution.

4 Applications

Example 4.1 Let $X = \mathbf{R}^2$. For the vector $x = (x_1, x_2)^T \in \mathbf{R}^2$, we define

$$\|x\| = \sum_{i=1}^2 |x_i|.$$

For the matrix $A = (a_{ij})_{2 \times 2}$, we define

$$\|A\| = \max_{1 \leq j \leq 2} \sum_{i=1}^2 |a_{ij}|.$$

We consider the following fractional-order neural network model with finite delay (FNND) on X :

$$\begin{cases} {}^c D_t^{\frac{2}{3}} x_1(t) = -2x_1(t) + \frac{\sin 2\pi t}{10(t+t^{\frac{1}{2}})^{\frac{1}{3}}} [3x_1(t+\theta) + x_2(t+\theta) + 3], & t > 0, \\ {}^c D_t^{\frac{2}{3}} x_2(t) = -2x_2(t) + \frac{\cos 2\pi t}{10(t+t^{\frac{1}{2}})^{\frac{1}{3}}} [x_1(t+\theta) + 2x_2(t+\theta) + 4], & t > 0, \\ x_1(\theta) = x_2(\theta) = 0.1, & -1 \leq \theta \leq 0. \end{cases} \quad (4.1)$$

Problem (4.1) can be written in the vector form as follows:

$$\begin{aligned} {}^c D_t^{\frac{2}{3}} x(t) &= Bx(t) + F(t, x_t), \quad t \geq 0, \\ x(t) &= 0.1, \quad t \in [-1, 0], \end{aligned}$$

where

$$\begin{aligned} x(t) &= (x_1(t), x_2(t))^T, \\ B &= \text{diag}\{-2, -2\}, \\ F(t, x_t) &= A(t)f(x_t) + C(t), \\ A(t) &= (a_{ij}(t))_{2 \times 2} = \begin{pmatrix} \frac{3 \sin 2\pi t}{10(t+t^{\frac{5}{2}})^{\frac{1}{3}}} & \frac{\sin 2\pi t}{10(t+t^{\frac{5}{2}})^{\frac{1}{3}}} \\ \frac{\cos 2\pi t}{10(t+t^{\frac{5}{2}})^{\frac{1}{3}}} & \frac{\cos 2\pi t}{5(t+t^{\frac{5}{2}})^{\frac{1}{3}}} \end{pmatrix}, \\ f(x_t) &= (x_{1t}, x_{2t})^T, \\ C(t) &= \left(\frac{3 \sin 2\pi t}{10(t+t^{\frac{5}{2}})^{\frac{1}{3}}}, \frac{2 \cos 2\pi t}{5(t+t^{\frac{5}{2}})^{\frac{1}{3}}} \right)^T. \end{aligned}$$

It is well known ([31]) that B generates a bounded operator semigroup

$$T(t) = e^{Bt} = \text{diag}\{e^{-2t}, e^{-2t}\}$$

and

$$\|T(t)\| \leq e^{-2t}, \quad t \geq 0$$

(i.e. $M = 1$, $\mu = 2$). Moreover, (3.1) is now the solution of (4.1) (Remark 3.2(iii)).

For $\varphi, \tilde{\varphi} \in C([-1, 0], X)$, it is easy to see that

$$\|F(t, \varphi) - F(t, \tilde{\varphi})\| \leq \frac{2}{5(t+t^{\frac{5}{2}})^{\frac{1}{3}}} \|\varphi - \tilde{\varphi}\|_{[-1, 0]} := L_f(t) \|\varphi - \tilde{\varphi}\|_{[-1, 0]}.$$

Since

$$\begin{aligned} (t+t^{\frac{5}{2}})^{-\frac{1}{3}} &\leq t^{-\frac{1}{3}}, \\ \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds &= \pi, \\ \int_0^\infty \frac{dt}{1+t^\alpha} &= \frac{\pi}{\alpha \sin(\pi/\alpha)} \quad (\alpha \in (1, 2)), \end{aligned}$$

we deduce that $(t+t^{\frac{5}{2}})^{-\frac{1}{3}} \in L^1(0, t)$ and

$$\begin{aligned} &\int_0^t (t-s)^{-\frac{1}{3}} L_f(s) ds \\ &= \frac{2}{5} \int_0^t (t-s)^{-\frac{1}{3}} s^{-\frac{1}{3}} \cdot (1+s^{\frac{3}{2}})^{-\frac{1}{3}} ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{2}{5} \left(\int_0^t [(t-s)^{-\frac{1}{3}} s^{-\frac{1}{3}}]^{\frac{3}{2}} ds \right)^{\frac{2}{3}} \cdot \left(\int_0^t ((1+s^{\frac{3}{2}})^{-\frac{1}{3}})^3 ds \right)^{\frac{1}{3}} \\ &\leq \frac{2\pi}{5} \left(\frac{4}{3\sqrt{3}} \right)^{\frac{1}{3}}. \end{aligned}$$

Hence

$$\frac{1}{\Gamma(\frac{2}{3})} \sup_{t \geq 0} \int_0^t \frac{L_f(s)}{(t-s)^{\frac{1}{3}}} ds \leq \frac{2\pi}{5\Gamma(\frac{2}{3})} \left(\frac{4}{3\sqrt{3}} \right)^{\frac{1}{3}} \approx 0.85 < 1.$$

Moreover,

$$\begin{aligned} \|F(t, 0)\| &\leq \frac{7}{10(t+t^{\frac{5}{2}})^{\frac{1}{3}}}, \\ \sup_{t \geq 0} \int_0^t (t-s)^{-\frac{1}{3}} \|F(s, 0)\| ds &\leq \frac{7\pi}{10} \left(\frac{4}{3\sqrt{3}} \right)^{\frac{1}{3}} < \infty \end{aligned}$$

and

$$\begin{aligned} \|F(t+1, \varphi) - F(t, \varphi)\| &\leq \left[\frac{1}{((t+1) + (t+1)^{\frac{5}{2}})^{\frac{1}{3}}} + \frac{1}{(t+t^{\frac{5}{2}})^{\frac{1}{3}}} \right] \cdot \left(\frac{2}{5} \|\varphi\|_{[-\delta, 0]} + \frac{7}{10} \right) \\ &\rightarrow 0, \quad t \rightarrow \infty. \end{aligned}$$

Therefore, the conditions in Theorem 3.5 are satisfied. Thus, by virtue of Theorem 3.5, the problem (4.1) has a unique S-asymptotically 1-periodic solution. We refer the reader to Figure 1 below for a numerical solution of (4.1).

Example 4.2 Let $X = \mathbf{R}$. We consider the following fractional-order model on X :

$$\begin{cases} {}^c D_t^{\frac{1}{2}} x(t) = \frac{\cos x(t+\theta)}{4(t+t^4)^{\frac{1}{4}}} + \sin 2\pi t, & t > 0, \\ x(\theta) = 1, & -1 \leq \theta \leq 0. \end{cases} \quad (4.2)$$

Problem (4.2) can be written in the form as follows:

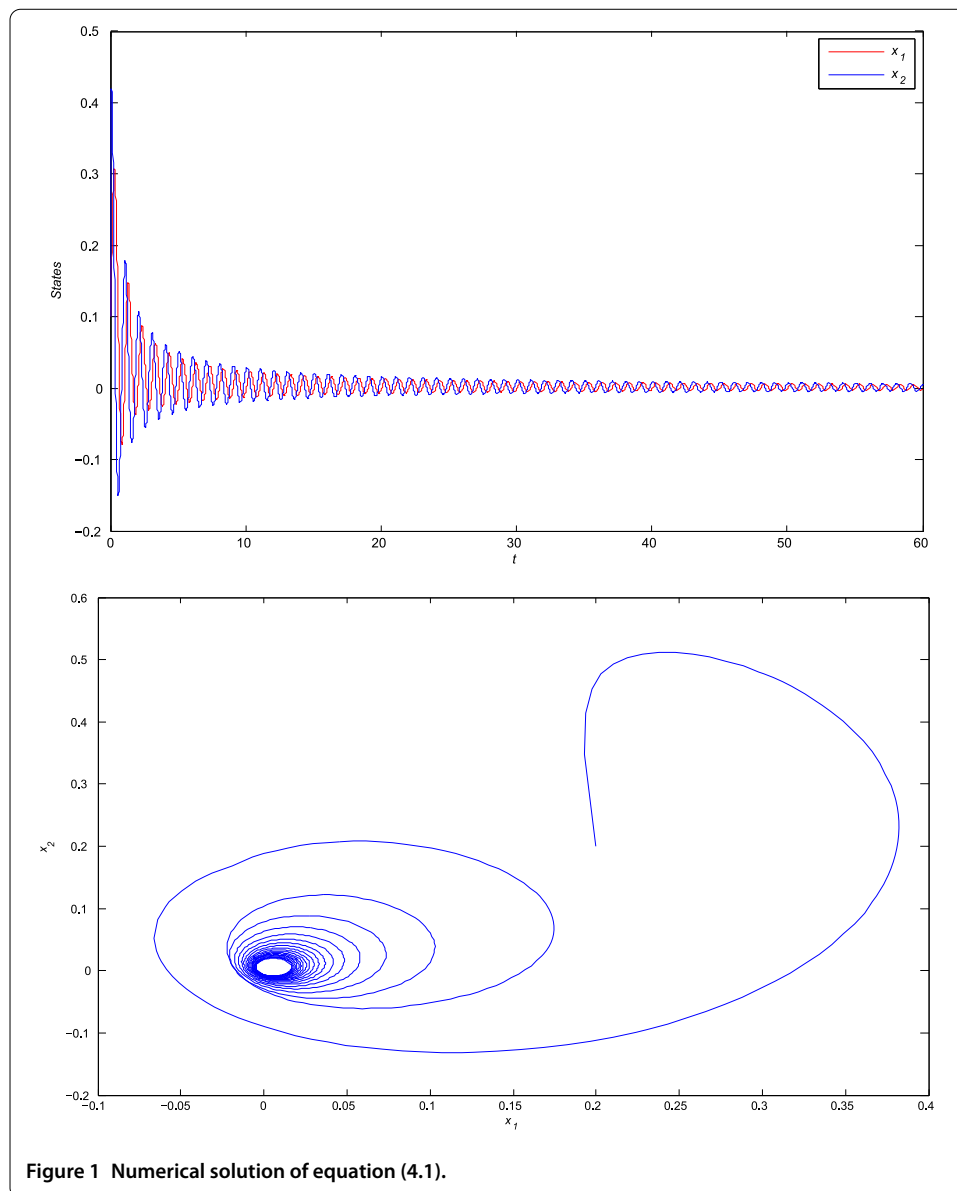
$$\begin{aligned} {}^c D_t^{\frac{1}{2}} x(t) &= F(t, x_t), \quad t > 0, \\ x(t) &= 1, \quad t \in [-1, 0], \end{aligned}$$

where

$$F(t, x_t) = \frac{\cos x_t}{4(t+t^4)^{\frac{1}{4}}} + \sin 2\pi t.$$

For any $\varphi, \tilde{\varphi} \in C([-1, 0], X)$, we have

$$\|F(t, \varphi) - F(t, \tilde{\varphi})\| \leq \frac{1}{4(t+t^4)^{\frac{1}{4}}} \|\varphi - \tilde{\varphi}\|_{[-1, 0]} := L_f(t) \|\varphi - \tilde{\varphi}\|_{[-1, 0]}.$$

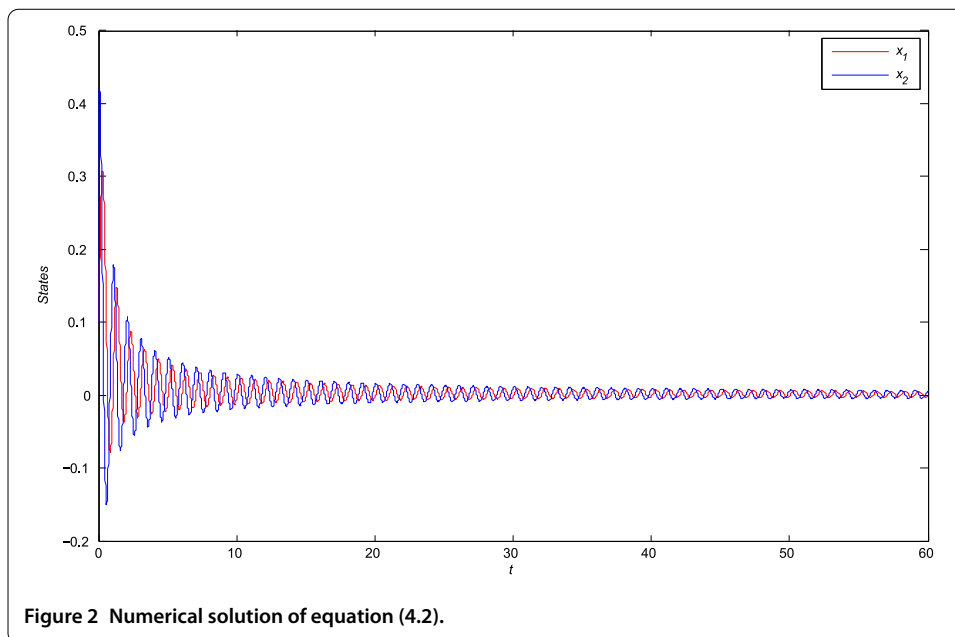


Noting that

$$(t + t^4)^{-\frac{1}{4}} \leq t^{-\frac{1}{4}}, \quad \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds = \pi,$$

we get

$$\begin{aligned} \int_0^t (t-s)^{-\frac{1}{2}} L_f(s) ds &= \frac{1}{4} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{4}} \cdot (1+s^3)^{-\frac{1}{4}} ds \\ &= \frac{1}{4} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \cdot s^{\frac{1}{4}} (1+s^3)^{-\frac{1}{4}} ds \\ &\leq \frac{\pi}{4}. \end{aligned} \quad (4.3)$$



So

$$\frac{1}{\Gamma(\frac{1}{2})} \sup_{t \geq 0} \int_0^t \frac{L_f(s)}{(t-s)^{\frac{1}{2}}} ds \leq \frac{\sqrt{\pi}}{4} \approx 0.44 < 1.$$

Moreover,

$$F(t, 0) = \frac{1}{4(t+t^4)^{\frac{1}{4}}} + \sin 2\pi t,$$

$$\int_0^t (t-s)^{-\frac{1}{2}} \sin 2\pi s ds = 2 \sin 2\pi t \int_0^{\sqrt{t}} \cos 2\pi s^2 ds - 2 \cos 2\pi t \int_0^{\sqrt{t}} \sin 2\pi s^2 ds.$$

In view of

$$\int_0^{+\infty} \sin s^2 ds = \int_0^{+\infty} \cos s^2 ds = \sqrt{\frac{\pi}{8}}$$

being Fresnel integrals and (4.3), we obtain

$$\sup_{t \geq 0} \left| \int_0^t (t-s)^{-\frac{1}{2}} F(s, 0) ds \right| < 2.$$

Clearly,

$$\lim_{t \rightarrow \infty} \left(\frac{t}{1+t^3} \right)^{\frac{1}{4}} = 0.$$

Hence, for each $\varepsilon > 0$, there is a positive constant $L > 0$ such that

$$\left(\frac{t}{1+t^3} \right)^{\frac{1}{4}} \leq \varepsilon, \quad t > L.$$

Therefore, noting that $t - L > \frac{t}{L}(L - s)$, we obtain

$$\begin{aligned} & \int_0^t (t-s)^{-\frac{1}{2}} \|F(s+1, \varphi) - F(s, \varphi)\| ds \\ & \leq \frac{1}{4} \int_0^L (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \left[\left(\frac{s}{1+s^3} \right)^{\frac{1}{4}} + \left(\frac{s}{1+s} \right)^{\frac{1}{4}} \left(\frac{s}{1+(s+1)^3} \right)^{\frac{1}{4}} \right] ds \\ & \quad + \frac{1}{4} \int_L^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \left[\left(\frac{s}{1+s^3} \right)^{\frac{1}{4}} + \left(\frac{s}{1+s} \right)^{\frac{1}{4}} \left(\frac{s}{1+(s+1)^3} \right)^{\frac{1}{4}} \right] ds \\ & \leq \frac{1}{2} \left(\frac{L}{t} \right)^{\frac{1}{2}} \int_0^L (L-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds + \frac{\varepsilon}{2} \int_L^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \\ & \leq \frac{\pi}{2} \left[\left(\frac{L}{t} \right)^{\frac{1}{2}} + \varepsilon \right], \end{aligned}$$

i.e. (H3') is satisfied. Thus, the conditions in Theorem 3.7 are fulfilled. Hence, by Theorem 3.7, the problem (4.2) has a unique S-asymptotically 1-periodic solution. We refer the reader to Figure 2 above for a numerical solution of (4.2).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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